p-Adic TGD: Mathematical ideas.

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Abstract

The mathematical basis of p-adic Higgs mechanism discussed in papers hep-th@xxx.lanl.gov 9410058-62 is considered in this paper. The basic properties of p-adic numbers, of their algebraic extensions and the so called canonical identification between positive real numbers and p-adic numbers are described. Canonical identification induces p-adic topology and differentiable structure on real axis and allows definition of definite integral with physically desired properties. p-Adic numbers together with canonical identification provide analytic tool to produce fractals. Canonical identification makes it possible to generalize probability concept, Hilbert space concept, Riemannian metric and Lie groups to p-adic context. Conformal invariance generalizes to arbitrary dimensions since p-adic numbers allow algebraic extensions of arbitrary dimension. The central theme of all developments is the existence of square root, which forces unique quadratic extension having dimension $D=4$ and $D=8$ for $p>2$ and $p=2$ respectively. This in turn implies that the dimensions of p-adic Riemann spaces are multiples of 4 in $p>2$ case and of 8 in $p=2$ case.

Note:

The .eps files representing p-adic fractals discussed in the text as well as MATLAB programs needed to generate the fractals are supplied by request. The commands attaching .eps files to text are in the text but preceided by comment signs: please remove these signs.

1 Introduction

There are a lot of speculations about the role of p-adic numbers in Physics [Volovich, Freund and Olson, Gervais]. In [Brekke and Freund] one can find a review of the work done. In general the work is related to quantum theory and based on assumption that quantum mechanical state space is ordinary complex Hilbert space. This is not absolutely necessary since p-adic unitarity and probability concepts make sense [Khrennikov]. What is however essential is some kind of correspondence between p-adic and real numbers since the predictions of, say, p-adic quantum mechanics should be expressed in terms of real numbers. The formulation of physical theory using p-adic state space and p-adic dynamical variables requires also the construction of p-adic differential and integral calculus. Also the p-adic counterpart of Riemann geometry as well as group theory are needed. In this chapter the aim is to carry out these generalizations.

The key observation behind all developments to be represented in the sequel is very simple: there is canonical correspondence between p-adic numbers and nonnegative real numbers given by "pinary" expansion of real number: positive real number $x = \sum x_n p^n$ ($x = 0, 1, .., p-1$, p prime) is mapped to p-adic number $\sum x_n p^{-n}$. This canonical correspondence allows to induce p-adic topology and differentiable structure to the real axis. p-Adically differentiable functions define typically fractal like real functions via canonical identification so that p-adic numbers provide analytic tool for producing fractals. p-adic numbers allow algebraic extensions of arbitrary dimension and the concept of complex analyticity generalizes to p-adic analyticity. The fact that real continuity implies p-adic continuity implies that real physics can emerge above some length scale L_p as an excellent approximation of underlying p-adic physics.

The canonical correspondence makes possible to generalize the concepts of inner product, integration, Hilbert space, Riemannian metric, Lie group theory and Quantum mechanics to p-adic context in a relatively straightforward manner. Essentially the fractal counterparts of all these structures are obtained in this manner. A possible reason for the practical absence of p-adic physics is probably that the existence and importance of the canonical correspondence has not been realized. The successfull p-adic description of Higgs mechanism relies heavily on canonical correspondence. In later chapters it

will be found that the concepts of p-adic probability and unitarity make sense and one can associate with p-adic probabilities unique real probabilities using canonical correspondence and this predicts novel physical effects.

The topics of the chapter are following:
i) p-Adic numbers, their algebraic extensions and canonical identification are described. The existence of square root of p-adically real number is necessary in many applications of p-adic numbers (p-adic group theory, p-adic unitarity, Riemannian geometry) and its existence implies unique algebraic extension, which is 4-dimensional in $p > 2$ case and 8-dimensional in $p = 2$ case.
ii) p-Adic valued inner product necessary for various generalizations is introduced.
iii) The concepts of p-adic differentiability and analyticity are introduced and the fractal properties of p-adically differentiable functions as well as nondeterminism of p-adic differential equations are demonstrated. It is also shown that period doubling property is characteristic feature of 2-adically differentiable functions.
iii) The concept of p-adic valued integration is defined: this concept is necessary in order to formulate p-adic variation principles.
iv) formulate p-adic Riemannian needed in TGD: eish applications: the existence of p-adic inner product and p-adic valued integration is essential for these developments. The dimensions of p-adic Riemann spaces are multiples of 4 ($p > 2$) or 8 ($p = 2$). It is hardly an accident that these dimensions are spacetime and imbedding space dimensions in TGD.
v) consider some characteristic details related to p-adic counterparts of Lie-groups

2 p-Adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of rational numbers using norm which, is different from the ordinary norm of real numbers [Borevich and Shafarevich]. p-Adic numbers are representable as power expansion of the prime number p of form:

$$x = \sum_{k \geq k_0} x(k)p^k, \; x(k) = 0, ..., p-1 \tag{1}$$

The norm of a p-adic number given by

$$|x| = p^{-k_0(x)} \tag{2}$$

Here $k_0(x)$ is the lowest power in the expansion of p-adic number. The norm differs drastically from the norm of ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \tag{3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of complex number.

The distance function $d(x, y) = |x - y|_p$ defined by p-adic norm possesses a very general property called ultrametricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\} \tag{4}$$

The properties of the distance function make it possible to decompose the R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x, y) \leq D \tag{5}$$

This division of the metric space into classes has following properties:
a) Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

b) Distances of points x and y inside single class are smaller than distances between different classes.
c) Classes form a hierarchical tree.

Notice that the concept of ultrametricity emerged to Physics in the models for spin glassess and is believed to have also applications in biology [Parisi]. The emergence of p-adic topology as effective topology of spacetime would make ultrametricy property basic feature of Physics at long length scales.

3 Canonical correspondence between p-adic and real numbers

There exists a natural continuous map $Id : R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \\ y_k &\in \{0, 1, .., p-1\} \end{aligned} \tag{6}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also desimal expansion is not unique ($1 = 0.999...$) for real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \end{aligned} \tag{7}$$

The p-adic images associated with these expansions are different

$$y_1 = \sum_{k=N_0}^{N} x_k p^k$$

$$
\begin{aligned}
y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\
&= y_1 + (x_N - 1)p^N - p^{N+1} \qquad (8)
\end{aligned}
$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is natural choice since in applications one always must use pinary cutoff in real axis. Furthermore. p-adicity is good approximation only below (rather than above, as thought originally) some length scale, which means pinary cutoff on real axis.

What about the p-adic counterpart of negative real numbers? In TGD:eish applications this correspondence is not needed since canonical identification is used only in the direction $R_p \rightarrow R$. Furthermore, it is always possible to choose the real coordinates of finite spacetime region so that coordinate variables are nonnegative so that the problem disappears.

The canonical identification map will be crucial for the proposed applications of p-adic numbers. The topology induced by this map in the set of positive real numbers differs from ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. **??**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above given length scale. This hierarchical ordering of p-adic topologies will be central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from p-adic topology is rougher than ordinary norm. This means that physical system can be genuinely p-adic below certain length scale L_p and become in good approximation real, when length scale resolution L_p is used in its description. TGD:eish applications rely on this assumption. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of p-adic topology.

The linear structure of p-adic numbers induces corresponding structure in the set of positive real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of real numbers. p-Adic multiplication is equivalent with ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace ordinary linearity in strongly nonlinear systems so that nonlinear systems would look simple in p-adic topology.

Canonical correspondence is quite essential in TGD:eish applications. Canonical identification makes it possible to define p-adic valued definite integral and this is cornerstone of TGD:eish applications. Canonical identification is in key role in the successfull predictions of the elementary particle masses. Canonical identification make also possible to undestand the connection between p-adic and real probabilities. These and many other successfull applications suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \\ |x|_p \leq (xy)_R &\leq x_R y_R \end{aligned} \tag{9}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to case of $(R_p)^n$ (linear space over p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \\ |\lambda|_p |y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \end{aligned} \tag{10}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \tag{11}$$

These inequilities resemble those satisfied by vector norm. The only difference is failure of linearity in the sense that the norm of scaled vector is not obtained by scaling the norm of original vector. Ordinary situation prevails only if scaling corresponds to power of p. Amusingly, the p-adic counterpart of Minkowskian norm

$$(x_R)^2 = (\sum_k x_k^2 - \sum_l x_l^2)_R \tag{12}$$

produces nonnegative norm. Clearly the p-adic space with this norm is analogous to future light cone.

These observations suggests that the concept of normed space or Banach space might have generalization and physically the generalization might apply to the description of nonlinear system. The nonlinearity would be concentrated in the nonlinear behaviour of the norm under scaling.

4 Algebraic extensions of p-adic numbers

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible [Borevich and Shafarevich]. The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in R_p assumed to have rational coefficients: irreduciblity means that polynomial does not possess roots in R_p so that one cannot decompose it into a product of lower order R_p valued polynomials. Denoting one of the roots of $P_n(t)$ by θ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k \theta^k \tag{13}$$

Since θ is root of the polynomial in R_p it follows that θ^n is expressible as sum of lower powers of θ so that these numbers indeed form n-dimensional linear space with respect to the p-adic topology.

Especially simple odd dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$\begin{aligned} P_n(t) &= t^n + \epsilon d \\ \epsilon &= \pm 1 \end{aligned} \tag{14}$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if d does not possess n:th root in R_p. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case θ is one of the roots of the equation

$$t^n = \pm d \tag{15}$$

where d is p-adic integer with finite number of pinary digits. It is possible although not necessary to identify roots as complex numbers. There exists n complex roots of d and θ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$\begin{aligned} \theta &= d^{1/n} exp(i\phi(m)), \ m = 0, 1, ..., n-1 \\ \phi(m) &= \frac{m2\pi}{n} \ or \ \frac{m\pi}{n} \end{aligned} \tag{16}$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of p-adic numbers with complex numbers plays fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

The norm of an algebraically extended p-adic number x can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \rightarrow^e R_p^n$ defined by the multiplication with x: $y \rightarrow xy$

$$N(x) = |det(x)|^\alpha, \ \alpha > 0 \tag{17}$$

The requirement that norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of R^n) implies the condition $\alpha = 1/n$, where n is the dimension of the algebraic extension.

The canonical correspondence between the points of R_+ and R_p generalizes in obvious manner: the point $\sum_k x_k\theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, ..,)$ of R^n using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces linear structure in R_+^n and p-adic multiplication induces multiplication for the vectors of R_+^n. An exciting possibility is that p-adic linearity might replace ordinary linearity in strongly nonlinear systems.

5 Algebraic extension allowing square root on p-adic real axis

The existence of square root of real p-adic number is a common theme in various applications of p-adic numbers.
a) The p-adic generalization of the representation theory of ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of p-adic numbers allowing square roots of real p-adic numbes is used. The reason is that matrix elements of the raising and lowering operators in Lie-algebras as well as oscillator operators typically involve square roots.
b) The existence of square root of real p-adic number is also necessary ingredient in the definition of p-adic unitarity and quantum probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is p-adically real leads to expressions involving square roots.
c) p-Adic Riemannian geometry necessitates the existence of square root of real p-adic numbers since the definition of the infinitesimal length $ds = \sqrt{g_{ij}dx^idx^j}$ involves square root.
What is important is that only the square root of p-adically real numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find finite dimensional extension allowing square root for all numbers of the extension. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.
a) For $p > 2$ p-adic number y in the range $(0, p-1)$ allows square root only provided there exists p-adic number $x \in \{0, p-1\}$ satisfying the condition $y = x^2 \ mod \ p$. Let x_0 be the smallest integer, which does not possess p-adic square root and add the square root θ of x_0 to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p-1\}$ as is easy to see. p-adic numbers $mod \ p$ form a finite field $G(p, 1)$ [Borevich and Shafarevich] so that any p-adic number y, which does not possess square root can be written in the form $y = x_0 u$, where u possesses square root. Since θ is by definition the square root of x_0 then also y possesses square root. The extension does not depend on the choice of x_0.

The square root of -1 does not exist for $p \, mod \, 4 = 3$ [Allenby and Redfern] and $p = 2$ but the addition of θ gurantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by θ. $\sqrt{(-1+p)-p}$ can be developed in power in powers of p and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p-1$ doesn not possess square one can take θ to be equal to $\sqrt{-1}$.

b) The next step is to add square root of p so that extension becomes 4-dimensional and arbitrary number in the extension can be written as

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \tag{18}$$

This extension is natural for p-adication of spacetime surface so that spacetime can be regarded as a number field locally. An important point to notice that the extension guarantees the existence of square for real p-adic numbers only.

c) In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since n:th Taylor coefficient is proportional to 2^{-n} and possesses 2-adic norm 2^n. If x_0 possesses norm 1 then x_1 must possess norm smaller than $1/8$ for series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one

obtains 8-dimensional extension. In TGD imbedding space $H = M^4_+ \times CP_2$ can be regarded locally as 8-dimensional extension of p-adic numbers. It is probably not an accident that the dimensions of minimal extensions allowing square roots are the space time and imbedding space dimensions of TGD.

By construction any p-adically real number in the extension allows square root. The square root for an arbitrary number sufficiently near real axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ in point of p-adic real axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to R^4 and the points of extension allowing square root form a set consisting of disjoint converge cubes of square root function forming structure resembling lightcone.

5.1 p-Adic square root function for $p > 2$

The study of the properties of the series representation of square root function shows that the definition of square root function is possible in certain region around real p-adic axis. What is nice that this region can be regarded as the p-adic counterpart of the future light cone defined by the condition

$$N_p(Im(Z)) \quad < \quad N_p(t = Re(Z)) = p^k \tag{19}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as time coordinate and the imaginary part of the p-adic coordinate is identified as spatial coordinate. p-Adic norm for four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$: at points $t = p^k$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to parallelpiped rather than ball.

The result can be understood heuristically as follows.
a) For four-dimensional extension allowing square root ($p > 2$) one can construct square root at each p-adically real point $x(k,s) = sp^k$, $s = 1, ..., p-1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.
b) Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

$$N_p(Z - sp^k) \leq R(k) \tag{20}$$

and having radius $R(k) = p^d, d \in Z$ around the expansion point.
c) A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultrametricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function only in the union of p-adic convergence spheres associated with the p-adically real points $x(k,s) = sp^k$.
d) By the scaling properties of the square root function the convergence radius $R(x(k,s)) \equiv R(k)$ is related to $R(x(0,s)) \equiv R(0)$ by the scaling factor p^{-k}:

$$R(k) = p^{-k}R(0) \tag{21}$$

so that convergence sphere expands as a function of p-adic time coordinate. The study of convergence reduces to the study of the series at points $x = s = 1, ..., k-1$ with unit p-adic norm.
e) Two neighbouring points $x = s$ and $x = s+1$ cannot belong to same convergence sphere: this would lead to contradiction with basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

$$R(0) < 1 \tag{22}$$

The requirement that convergence is achieved at all points of the real axis implies

$$\begin{aligned} R(0) &= \frac{1}{p} \\ R(p^k s) &= \frac{1}{p^{k+1}} \end{aligned} \tag{23}$$

If the convergence radius is indeed this then the region, where square root is defined corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is p-adic counterpart of M^4 light cone. If convergence radius is smaller the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$\begin{aligned} N_p(Z) &= (N_p(det(Z)))^{\frac{1}{4}} \\ &= (N_p((x^2+y^2)^2 + 2p^2((xv-yu)^2 + (xu-yv)^2) + p^4(u^2+v^2)^2))^{\frac{1}{4}} \end{aligned} \tag{24}$$

where $det(Z)$ is the determinant of the linear map defined by multiplication with Z. The definition of convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2+v^2) + p^4(u^2+v^2)^2)) < 1 \tag{25}$$

For physically interesting case $p \ mod \ 4 = 3$ the points (y, u, v) satisfying the conditions

$$\begin{aligned} N_p(y) &\leq \frac{1}{p} \\ N_p(u) &\leq 1 \\ N_p(v) &\leq 1 \end{aligned} \tag{26}$$

belong to the sphere of convergence: it is essential that for all u and v satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers the real counterpart of the sphere $r = t$ is now parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with average velocity of light in discrete steps at times $t = p^k$.

The emergence of p-adic light cone as a natural p-adic coordinate space is in nice accordance with the basic assumptions about the imbedding space of TGD and shows that big bang cosmology might basically related to the

existence of p-adic square root! The result gives also support for the idea that p-adicity is responsible for the generation of lattice structures (convergence region for any function is expected to be more or less parallelpiped like region).

A peculiar feature of the p-adic light cone is the instantaneous expansion of 3-space at moments $t_p = p^k$. A possible physical interpretation is that p-adic light cone or rather the individual convergence cube of the light cone represents the time development of single maximal quantum coherent region at p-adic level of topological condensate (probably there are many of them). The instantaneous scaling of the size of region by factor $\sqrt{p}$ at moment $t_R = p^{k/2}$ corresponds to a phase transition and thus to quantum jump. Experience with p-adic QFT indeed shows that $L_p = \sqrt{p}L_0$,$L_0 \simeq 10^4\sqrt{G}$ appears as infrared cutoff length for the p-adic version of standard model so that p-adic continuity is replaced with real continuity (implying p-adic continuity) above the length scale L_p. The idea that larger p-adic length scales $p^{k/2}L_p$, $k > 0$, would form quantum coherent regions for physically most interesting values of p is probably unrealistic and L_p probably gives a typical size of 3-surface at p:th condensate level. Of course, already this hypothesis is far from trivial since L_p can have arbitrarily large values so that arbitrarily large quantum coherent systems would be possible.

$p = M_{127}$, the largest physically interesting Mersenne prime, provides an interesting example:
i) $p = M_{127}$-Adic light cone does not make sense for time and length scales smaller than length scale defined by electron Compton length and QFT below this length scale makes sense.
ii) The first phase transition would happen at time of order 10^{-1} seconds, which corresponds to length scales of order 10^7 meters.
iii) The next phase transition takes place at $t_R \simeq 10^{11}$ light years and corresponds to the age of the Universe.

Recent work with the p-adic field theory limit of TGD has shown that the convergence cube of p-adic square root function having size $L_p = \frac{L_0}{\sqrt{p}}$, $L_0 = 1.824 \cdot 10^4\sqrt{G}$, serves as a natural quantization volume for p-adic counterpart of standard model. An open question is whether also larger convergence cubes serve as quantization volumes or whether L_p gives natural upper bound for the size of p-adic 3-surfaces. The original idea that p-adic manifolds,

constructed by gluing together pieces of p-adic light cone together along their sides, could be used to build Feynmann graphs with lines thickened to 4-manifolds has turned out to be not useful for physically most interesting (large) values of p.

5.2 Convergence radius for square root function

In the following it will be shown that the convergence radius of $\sqrt{t+Z}$ is indeed nonvanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t+Z}$ reads as

$$\begin{aligned}
\sqrt{t+Z} &= = \sqrt{x}\sum_n a_n \\
a_n &= (-1)^n \frac{(2n-3)!!}{2^n n!} x^n \\
x &= \frac{Z}{t}
\end{aligned} \tag{27}$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \rightarrow \infty$. The p-adic norm of n:th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{n!}) N_p(x^n) < N_p(x^n) N_p(\frac{1}{n!}) \tag{28}$$

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \ for \ \ N_p(x) < 1 \tag{29}$$

holds true. The strategy is as follows:
a) The norm of x^n can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for U is derived at the limit of large n.

5.2.1 p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \leq n(i) < p$ be the p-adic expansion of n. Then $N_p(n!)$ can be expressed in the form

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \\ N(1) &= \frac{1}{p} \\ N(i+1) &= N(i)^{p-1}p^{-i} \end{aligned} \tag{30}$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \tag{31}$$

Proof: $n!$ can be written as a product

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} X(i, n(i)) \\ X(k, n(k)) &= N_p((n(k)p^k)!) \\ X(k-1, n(k-1)) &= N_p(\prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k + i)) = N_p((n(k-1)p^{k-1})!) \\ X(k-2, n(k-2)) &= N_p(\prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k + n(k-1)p^{k-1} + i)) \\ &= N_p((n(k-2)p^{k-2})!) \\ X(k-i, n(k-i)) &= N_p((n(k-i)p^{k-i})!) \end{aligned} \tag{32}$$

The factors $X(k, n(k))$ reduce in turn to the form

$$\begin{aligned} X(k, n(k)) &= \prod_{i=1}^{n(k)} Y(i, k) \\ Y(i, k) &= \prod_{m=1}^{p^k} N_p(ip^k + m) \end{aligned} \tag{33}$$

The factors $Y(i,k)$ in turn are indentical and one has

$$\begin{aligned} X(k,n(k)) &= X(k)^{n(k)} \\ X(k) &= N_p(p^k!) \end{aligned} \tag{34}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of k:
1) $X(1) = N_p(p!) = p^{-1}$
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ($p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to p^2.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$\begin{aligned} N_p(n!) &= p^{-\sum_{i=0}^{k} n(i)A(i)} \\ A(i) &= \sum_{m=1}^{i} m(p-1)^{i-m} \end{aligned} \tag{35}$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$\begin{aligned} A(i) &= (\frac{p-1}{p-2})^2(p-1)^{i-1}(1+\frac{i}{(p-1)^{i+1}}-\frac{(i+1)}{(p-1)^i}) \\ &\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \end{aligned} \tag{36}$$

5.2.2 Upper bound for $N_p(\frac{x^n}{n!})$ for $p > 2$

By using the expressions $n = \sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_pn!$ as well as the upper bound

$$A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \tag{37}$$

for $A(i)$ one obtains the upper bound

$$N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^{k} n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \tag{38}$$

It is clear that for $N_p(x) < 1$ that is $K \geq 1$ the upper bound goes to zero. For $p > 3$ exponents are negative for all values of i: for $p = 3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as majorant.

5.3 $p = 2$ case

In $p = 2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$N_p(a_n) \;=\; N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^{k} n(i)\frac{i(i+1)}{2^{i+1}}} \tag{39}$$

At the limit $n \rightarrow \infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K > 1$ so that one has

$$N_p(Z) \;\equiv\; (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \tag{40}$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$\sqrt{s+\frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \tag{41}$$

so that one can develop square as series in all half odd integer points of p-adic real axis. As a consequence the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Spacetime has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces). Since $\sqrt{Z}$ appears in

the definition of the fermionic Ramond fields one might wonder whether once could interpret this binary structure as a geometric representation of half odd integer spin. The coordinate space associated with spacetime representable as a four-dimensional subset of this light cone inherits the light cone structure.

5.4 p-Adic inner product and Hilbert spaces

Concerning the physical applications of complex p-adic numbers the problem is that p-adic norm is not bilinear in its arguments and therefore it does not define inner product and angle. One can however consider a generalization of the ordinary complex inner product $\bar{z}z$ to p-adic valued inner product. It turns out that p-adic quantum mechanics in the sense as it is used in p-adic TGD can be based on this inner product.

Restrict the consideration to minimal extension allowing square roots near real axis ($p > 2$) and denote the complex conjugate of Z with Z_c and by $\hat{Z}$ the conjugate of Z under the conjugation $\sqrt{p} \rightarrow -\sqrt{p}$: $Z \rightarrow \hat{Z} = x + \theta x - \sqrt{p}(u + \theta v)$. The inner product in the 4-dimensional extension of p-adic numbers reads as

$$\langle Z, Z \rangle = Z_c Z + \hat{Z}_c \hat{Z} = 2(x^2 + y^2 + p(u^2 + v^2)) \qquad (42)$$

This inner product is bilinear and symmetric, defines p-adically real norm and vanishes only if Z vanishes. This inner product leads to p-adic generalization of unitarity and probability concept. The solution of the unitarity condition $\sum_k S_{mk}\bar{S}_{nk} = \delta(m, n)$ involves square root operations and therefore the minimal extension for the Hilbert space is 4-dimensional in $p > 2$ case and 8-dimensional in $p = 2$ case. The physically most interesting consequences of this result are encountered in p-adic quantum mechanics.

The inner product associated with minimal extension allowing square root near real axis provides a natural generalization of the real and complex Hilbert spaces respectively. Instead of real or complex numbers square root allowing algebraic extension extension appears as the multiplier field of the Hilbert space and one can understand the points of Hilbert space as infinite sequences $(Z_1, Z_2, ..., Z_n, ...)$, where Z_i belongs to the extension. The inner product $\sum_k \langle Z_k^1, Z_2^k \rangle$ is completely analogous to the ordinary Hilbert space inner product.

A particular example of p-adic Hilbert space is obtained as a generalization of the space of complex valued functions $f : R^n \rightarrow C$. The inner product for functions f_1 and f_2 is just the previously defined inner product $\langle f_1(x), f_2(x) \rangle$ combined with integration over R_p^n: the definition of p-adic integration will be considered later in detail.

The inner product allows to define the concepts of length and angle for two vectors in p-adic extension possessing either p-adic or ordinary complex values. This implies that the concepts and of p-adic Riemannian metric, Kähler metric and conformal invariance become possible.

5.5 p-Adic Numbers and Finite Fields

Finite fields (Galois fields) consists of finite number of elements and allow sum, multiplication and division. A convenient representation for the elements of a finite field is as the roots of the polynomial equation $t^{p^m} - t = 0 \ mod \ p$, where p is prime, m an arbitrary integer and t is element of a field of characteristic p ($pt = 0$ for each t). The number of elements in finite field is p^m, that is power of prime number and the multiplicative group of a finite field is group of order $p^m - 1$. $G(p, 1)$ is just cyclic group Z_p with respect to addition and $G(p, m)$ is in rough sense m:th Cartesian power of $G(p, 1)$.

The elements of the finite field $G(p, 1)$ can be identified as the p-adic numbers $0, ..., p - 1$ with p-adic arithmetics replaced with modulo p arithmetics. Finite fields $G(p, m)$ can be obtained from m-dimensional algebraic extensions of p-adic numbers by replacing p-adic arithmetics with modulo p arithmetics. In TGD context only the finite fields $G(p > 2, 2)$, $p \ mod \ 4 = 3$ and $G(p = 2, 4)$ appear naturally. For $p > 2$, $p \ mod \ 4 = 3$ one has: $x + iy + \sqrt{p}(u + iv) \rightarrow x_0 + iy_0 \in G(p, 2)$.

As far as applications are considered the basic observation is that the unitary representations of p-adic scalings $x \rightarrow p^k x \ k \in Z$ lead naturally to finite field structures. These representations reduce to representations of finite cyclic group Z_m if $x \rightarrow p^m x$ acts trivially on representation functions for some value of m, $m = 1, 2, ...$ Representation functions, or equivalently the scaling momenta $k = 0, 1, ..., m - 1$ labeling them, have a structure of cyclic group. If $m \neq p$ is prime the scaling momenta form finite field $G(m, 1) = Z_m$ with respect to summation and multiplication modulo m.

The construction p-adic field theory shows that also the p-adic counterparts of ordinary planewaves carrying p-adic momenta $k = 0, 1..., p-1$ can be given the structure of Finite Field $G(p,1)$: one can also define complexified planewaves as square roots of the real p-adic planewaves to obtain Finite Field $G(p,2)$.

6 p-Adic differential calculus

It would be nice to have a generalization of the ordinary differential and integral calculus to p-adic case. Instead of trying to guess directly the formal definition of p-adic differentiability it is better to guess what kind of functions $f : R_p \rightarrow R$ might be natural candidates for p-adically differentiable functions and then try to find whether the concept of p-adic differentiability makes sense. There are several candidates for p-adically differentiable functions.

a) p-Analytic maps $R_p \rightarrow R_p$ representable as power series of p-adic argument induce via the canonical identification maps $R_+ \rightarrow R_+$. These maps are well defined for algebraic extensions of p-adic numbers, too and induce p-analytic maps $R_+^n \rightarrow R_+^n$ via the canonical correspondence. These functions correspond to ordered fractals.

b) A second candidate is obtained as a generalization of canonical identification map $R_p \rightarrow R$: $Y_D(x) = \sum x_k p^{-kD}$, where D is so call anomalous dimension. The corresponding map $R \rightarrow R$ is given by $\sum_k x_k p^{-k} \rightarrow \sum x_k p^{-kD}$: $D = 1$ gives identity map. These functions are not differentiable in the strict sense of the word and give rise to chaotic fractals, which resemble Brownian functions.

6.1 p-Analytic maps

p-analytic maps $g : R_p \rightarrow R_p$ satisfy the usual criterion of differentiability and are representable as power series

$$g(x) = \sum_k g_k x^k \tag{43}$$

Also negative powers are in principle allowed. The rules of p-adic differential

calculus are formally identical to those of the ordinary differential calculus and generalize in trivial manner for algebraic extensions.

The class of p-adically constant functions (in the sense that p-adic derivative vanishes) is larger than in real case: any function depending on finite number of positive pinary digits of p-adic number and of arbitrary number of negative pinary digits is p-adically constant. This becomes obvious, when one considers the definition of p-adic derivative: when the increment of p-adic coordinate becomes sufficiently small p-adic constant doesn't detect the variation of x since it depends on finite number of positive p-adic pinary digits only. p-adic constants correspond to real functions, which are constant below some length scale $\Delta x = 2^{-n}$. As a consequence p-adic differential equations are nondeterministic: integration constants are arbitrary functions depending on finite number of positive p-adic pinary digits. This feature is central as far applications are considered.

p-Adically analytic functions induce maps $R_+ \rightarrow R_+$ via the canonical identification map. The simplest manner to get some grasp on their properties is to plot graphs of some simple functions (see Fig. **??** for the graph of p-adic x^2 and for Fig. **??**) for the graph of p-adic $1/x$). These functions have quite characteristic features resulting from the special properties of p-adic topology:

a) p-Analytic functions are continuous and differentiable from right: this peculiar asymmetry is a completely general signature of p-adicity. As far as time dependence is considered the interpretation of this property as mathematical counterpart of irreversibelity looks natural. This suggests that the transition from reversible microscopic dynamics to irreversible macroscopic dynamics corresponds to the transition from the ordinary topology to effective p-adic topology.

b) There are large discontinuities associated with the points $x = p^n$. This implies characteristic theshold phenomena. Consider a system whose output $f(n)$ is function of input, which is integer n. For $n < p$ nothing peculiar happens but for $n = p$ the real counterpart of the output becomes very small for large values of p. In biosystems threshold phenomena are typical and p-adicity might be the key in their understanding. The discontinuities associated with powers of $p = 2$ are indeed encountered in many physical situations. Auditory experience has the property that given frequency ω_0 and its multiples $2^k\omega_0$, octaves, are experienced as same frequency suggesting the

auditory response function for a given frequency ω_0 is 2-adicallly analytic function. Titius-Bode law states that the mutual distances of planets come in powers of 2, when suitable unit of distance is used. In turbulent systems period doubling spectrum has peaks at frequencies $\omega = 2^k\omega_0$.
c) A second signature of p-adicity is "p-plicity" appearing in the graph of simple p-analytic functions. As an example, consider the graph of p-adic x^2 demonstrating clearly the decomposition into p steps at each interval $[p^k, p^{k+1})$.
d) The graphs of p-analytic functions are in general ordered fractals as the examples demonstrate. For example, power functions x^n are selfsimilar (the values of the function at some any interval (p^k, p^{k+1}) determines the function completely) and in general p-adic x^n with nonnegative (negative) n is smaller (larger) than real x^n expect at points $x = p^n$ as the graphs of p-adic x^2 and $1/x$ show (see Fig. **??**) These properties are easily understood from the properties of p-adic multiplication. Therefore the first guess for the behaviour of p-adically analytic function is obtained by replacing x and the coefficients g_k with their p-adic norms: at points $x = p^n$ this approximation is exact if the coefficients of the power series are powers of p. This step function approximation is rather reasonable for simple functions such as x^n as the figures demonstrate. Since p-adically analytic function can be approximated with $f(x) \sim f(x_0) + b(x - x_0)^n$ or as $a(x - x_0)^n$ (allowing nonanalyticity at x_0) around any point the fractal associated with p-adically analytic function has universal geometrical form in sufficiently small length scales.

p-Adic analyticity is well defined for the algebraic extensions of R_p, too. The figures **??** and **??** visualize the behaviour of real and imaginary parts of two adic z^2 function as function of real x and y coordinates in the parallelpiped I^2,$I = [1 + 2^{-7}, 2 - 2^{-7}]$. An interesting possibility is that the order parameters describing various phases of physical system are p-adically differentiable functions. The p-analyticity would therefore provide a means for coding the information about ordered fractal structures.

The order parameter could be one coordinate component of a p-adically analytic map $R^n \rightarrow R^n$, $n = 3, 4$. This is analogous to the possibility to regard the solution of Laplace equation in 2 dimensions as a real or imaginary part of an analytic function. A given region V of the order parameter space corresponds to a given phase and the volume of ordinary space occupied by

this phase corresponds to the inverse image $g^{-1}(V)$ of V. Very beautiful images are obtained if the order parameter is the the real or imaginary part of p-adically analytic function $f(z)$. A good example is p-adic z^2 function in the parallelpiped $[a,b]\times[a,b]$, $a=1+2^{-9}$, $b=2-2^9$ of C-plane. The value range of the order parameter can divided into, say, 16 intervals of same length so that each interval corresponds to a unique color. The resulting fractals possess features, which probably generalize to higher dimensional extensions.
a) The inverse image is ordered fractal and possessess lattice/ cell like structure, with the sixes of cells appearing in powers of p. Cells are however not identical in analogy with the differentiation of biological cells.
b) p-Analyticity implies the existence of local vector valued order parameter given by the p-analytic derivative of $g(z)$: the geometric structure of the phase portrait indeed exhibits the local orientation clearly.
c) In a given resolution there appear 0,1, and 2-dimensional structures and also defects inside structures. In 3-dimensional situation rather rich structures are to be expected.

Even more beautiful structures are obtained by adding some disorder: for instance, the composite map $z(x,y)=Y_D(x^2-y^2)$ for $D=1/2$ for $p=2$, where the function $Y_D(x)$ is defined in the next section gives rise to extremely beatiful fractal using the previous description. Noncolored pictures cannot reproduce the beauty of these fractals not suggested by the expectations based on the appearence of the graphs of $Y_D(x)$ (see Fig. **??**) (the MATLAB programs needed to generate p-adic fractals are supplied by request for interested reader).

These observations suggests that p-analyticity might provide a means to code the information about ordered fractal structures in the spatial behaviour of order parameters (such as entzyme concentrations in biosystems). An elegant manner to achieve this is to use purely real algebraic extension for 3-space coordinates and for the order parameter: the image of the order parameter $\Phi=\phi_1+\phi_2\theta+\phi_3\theta^2$ under the canonical identification is real and positive number automatically and might be regarded as concentration type quantity.

6.2 Functions Y_D

p-Analytic functions give rise to ordered fractals. One can find also functions describing disordered fractals. The simplest generalization of the identification of real and p-adic numbers to a chaotic fractal is the following one

$$Y_D(x_p) = \sum_n x(n) p^{-nD} \tag{44}$$

where D is constant. $D = 1$ gives identication map. Y_D defines in obvious manner a map $R_+ \rightarrow R_+$ via the canonical identification map. To each pinary digit there corresponds the power p^{-kD} so that a change of single pit induces change of form p^{-kD} nonlinear in the increment $|dx_p|$.

One can generalize the definition of Y_D . The anomalous dimension D can be p-adic constant and therefore depend on finite number of positive p-adic pinary digits of x_p and the most general definition of Y_D reads

$$\begin{aligned} Y_D(x_p) &= \sum_n x(n) p^{-nD(x_{<n+1})} \\ D &= D(x_p) \\ x_{<n} &= \sum_{k<n} x_k p^k \end{aligned} \tag{45}$$

Here it is essential to assume that the anomalous dimension associated with n:th pinary digit is the value of anomalous dimension associated with n:th pinary digit cutoff of x: otherwise p-adic continuity is lost. This generalization allows also fractal functions, which become ordinary smooth functions in sufficiently small length scales: the only assumption needed is that $D(x)$ approaches $D = 1$, when the number of pinary digits of x becomes large. The definition of the functions Y_D generalizes in trivial manner to higher dimensional case, the anomalous dimensions being now p-adically constant functions of all p-adic coordinates.

Although the functions Y_D are not differentiable in the strict sense of the word they have the property that if x_p has finite number of nonvanishing pinary digits then for sufficiently small increment dx_p so that x_p and dx_p have no common pinary digits one has just

$$Y_D(x_p + dx_p) - Y_D(x_p) \; = \; Y_D(dx_p) \tag{46}$$

$Y_D(dx_p)$ might be called D-differential with anomalous dimension D. This differential maps the tangent space of p-adic numbers to the tangent space of real numbers in fractal like manner in the sense that if p-adic and real tangent spaces are identified in canonical manner then D-differential induces nonlinear fractal like map of real tangent space to itself. The functions $Y_D(dx_p)$ are therefore good candidate for a fractal like generalization of linear differential dx_p.

The local anomalous dimension D corresponds to the so called Lifschitz-Hölder exponent α encountered in the theory of multifractals [Feder]. Multifractals are decomposed into union of fractals with various fractal dimensions by decomposing the range S of fractal function to a union $\cup_\alpha S_\alpha$ of disjoint sets S_α : S_α consists of points of S, for which the anomalous dimension D has fixed value α. One can associate to each set S_α its own fractal dimension and this decomposition plays important role in fractal analysis of the empirical data. The values of $D > 1$ and $D < 1$ one correspond to "antifractal" (ordinary derivative vanishes) and fractal (ordinary derivative is divergent) behavior respectively.

The simplest manner to see the fractality properties is to plot the graph of Y_D. The general features of the graph (see. Fig. **??**) are following:
a) Y_D is continuous from right and there are sharp discontinities associated with the points $x = p^m$. The graph of Y_D is selfsimilar if D is constant. The value of p reflects itself as a characteristic "p-peakedness" for $D < 1$ and "p-stepness" for $D < 1$.
b) For $D < 1$ Y_D is surjective but not injective. This is seen as typical fluctuating behaviour resembling that associated with Brownian motion. If $D < 1$ is constant there is infinite number of preimages associated with a given point y.
c) For $D > 1$ Y_D is constant almost everywhere, nonsurjective, and increases monotonically.
d) $x = 0$ and $x = 1$ are fixed points common to all Y_D. These points are attractors for $D < 1$ and repellors for $D > 1$. If $D < 1$ Y_D has also additional fixed points $x > 1$ in the neighbourhood $x = 1$.

The graph of Y_D, $D < 1$ resembles that of Brownian motion. The following arguments suggest that there is more than a mere analogy involved and that functions Y_D with p-adically constant D combined with ordinary differentiable functions might provide a description for random processess.
a) Y_D equals to p^{kD} at points $x = p^D$. For $D = 1/2$ this means that Y_D is analogous to the root mean square distance $d(t) = \sqrt{\langle r^2 \rangle}(t)$ from origin in Brownian motion, which behaves as $d \propto \sqrt{t}$.
b) In Brownian motion $d(t)$ is not differentiable function at origin: $d(t) \propto t^{1/2}$. The same holds true for Y_D,$D = 1/2$ at each point x so that Y_D in certain sense provides a simulation of Brownian motion.
c) Y_D is only the simplest example of Brownian looking motion and as such too simple to describe realistic situations. It is however possible to form composites of Y_D and p-adically differentiable functions as well as ordinary differentiable functions, which both are right differentiable with anomalous dimension $D = 1$. These functions contain also p-adic constants, which depend on finite number of pinary digits of t in arbitrary manner so that nondeterminism results. These features suggest that the functions Y_D provide basic element for the description of Brownian processess.
d) There is no obvious reason to exclude values of D different from $D = 1/2$ and this means that the concept of Brownian motion generalizes.
e) Since Brownian motion can be regarded as Gaussian process (the value of the increment of x obeys Gaussian distribution) it seems that also higher dimensional Gaussian processes possessing as their graphs Brownian surfaces could be described by using the higher dimensional algebraic extensions of p-adic numbers and corresponding higher dimensional extensions of Y_D. The deviation of D from $D = 1/2$ might correspond to anomalous dimensions deriving from the non-Gaussian behaviour implied by interactions.

One can form also functional composites of Y_D and anomalous dimensions are multiplicative in this process: $y_{D_1} \circ y_{D_2}$ possessess anomalous dimension $D = D_1 \times D_2$. For $D_i < 1$ the functional composition in general implies more chaotic behaviour. It must be emphasized, that functions Y_D (not very nice objects!) do not have appear in any applications of this book.

7 p-Adic integration

The concept of p-adic definite integral can be defined for functions $R_p \rightarrow C$ [Brekke and Freund] using translationally invariant Haar measure for R_p. In present context one is however interested in definining p-adic valued definite integral for functions $f: R_p \rightarrow R_p$: target and source spaces could of course be also some some algebraic extensions of p-adic numbers. What makes the definition nontrivial is that the ordinary definition as the limit of Riemann sum doesnt work: Riemann sum approaches to zero in p-adic topology and one must somehow circumvent this difficulty. Second difficulty is related to the absence of well ordering for p-adic numbers. The problems are avoided by defining integration essentially as the inverse of differentation and using canonical correspondence to define ordering for p-adic numbers.

The definition of p-adic integral functions defining integration as inverse of differentation operation is straightforward and one obtains just the generalization of standard calculus. For instance, one has $\int z^n = \frac{z^{n+1}}{(n+1)} + C$ and integral of Taylor series is obtained by generalizing this. One must however notice that the concept of integration constant generalizes: any function $R_p \rightarrow R_p$ depending on finite number of pinary digits only, has vanishing derivative.

Consider next definite integral. The absence of well ordering implies that the concept of integration range (a, b) is not well defined as purely p-adic concept. A possible resolution of the problem is based on canonical identification. Consider p-adic numbers a and b. It is natural to define a to be smaller than b if the canonical images of a and b satisfy $a_R < b_R$. One must notice that $a_R = b_R$ does not imply $a = b$ since the inverse of the canonical identification map is two-valued for real numbers having finite number of pinary digits. For two p-adic numbers a, b with $a < b$ one can define the integration range (a, b) as the set of p-adic numbers x satisfying $a \leq x \leq b$ or equivalently $a_R \leq x_R \leq b_R$. For a given value of x_R with finite number of pinary digits one has two values of x and x can be made unique by requiring it to have finite number of pinary digits.

One can define definite integral $\int_a^b f(x)dx$ formally as

$$\int_a^b f(x)dx \quad = \quad F(b) - F(a)$$

$$(47)$$

where $F(x)$ is integral function obtained by allowing only ordinary integration constants and $b_R > a_R$ holds true. One encounters however problem, when $a_R = b_R$ and a and b are different. Problem is avoided if integration limits are assumed to correspond p-adic numbers with finite number of pinary digits.

One could perhaps relate the possibility of p-adic integration constants depending on finite number of pinary digits to the possibility to decompose integration range $[a_R, b_R]$ as $a = x_0 < x_1 < ....x_n = b$ and to select in each subrange $[x_k, x_{k+1}]$ the inverse images of $x_k \leq x \leq x_{k+1}$, with x having finite number of pinary digits in two different manners. These different choices correspond to different integration paths and the value of the integral for different paths could correspond to the different choices of p-adic integration constant in integral function. The difference between a given integration path and 'standard' path is simply the sum of differences $F(x_k) - F(y_k)$, $(x_k)_R = (y_k)_R$.

This definition has several nice features:
a) Definition generalizes in obvious manner to higher dimensional case.
g) Standard connection between integral function and definite integral holds true and in higher dimensional case the integral of total divergence reduces to integral over boundaries of integration volume. This property guarantees that p-adic action principle leads to same field equations as its real counterpart. It this in fact this property, which drops other alternatives from consideration.
c) Integral is linear operation and additive as a set function.
d) The basic results of real integral calculus generalize as such to p-adic case.

There is however a problem related to the generalization of the integral to the case of non-analytic functions. For instance, the so called number theoretic plane waves defined as functions a^{kx} with $a \in \{1, p-1\}$ is so called primitive root satisfying $a^{p-1} = 1$ and $k \in Z$, are p-adic counterparts of ordinary plane waves and nonanalytic functions of x. The construction of field theory limit of TGD is based on Fourier analysis using p-adic planewaves. It is difficult to avoid the use of these functions in construction of p-adic version of perturbative QFT. What one needs is definition of integral guaranteing orthogonality of the p-adic plane waves in suitable integration range. A formal integration using the integration formula gives factor

$$\begin{aligned}\int_0^{p-1} a^{kx}dx &= \frac{1}{ln(a)}(a^{k(p-1)}-1)=0,\ k=1,...,p-1,\\ \int_0^{p-1} a^{kx}dx_{|k=0} &= \int_0^{p-1} dx = p-1 \end{aligned} \tag{48}$$

Although the factor $ln(a)$ is ill defined p-adically this does not matter since integral vanishes for $k \neq 0$: for $k = O$ the integral is in well defined. p-Adic planewaves are not differentiable in ordinary sense but the differentation can be defined purely algebraically as multiplication with p-adic momentum.

8 p-Adic manifold geometry

In the following the concepts of p-Adic Riemannian and conformal geometries are considered.

8.1 p-Adic Riemannian geometry

It is possible to generalize the concept of (sub)manifold geometry to p-adic (sub)manifold geometry. The formal definition of p-adic Riemannian geometry is based on p-adic line element $ds^2 = g_{kl}dx^kdx^l$. Lengths and angles are defined in the usual manner and their definition involves square root ds of the line element. The existence of square roots forces quadratic extension of p-adic numbers allowing square roots. As found the extension is 4-dimensional for $p > 2$ and 8-dimensional in $p = 2$ case. This extension in question must appear as coefficient ring of p-adic tangent space so that p-adic Riemann spaces must be locally cartesian powers of $4-$ $(p > 2)$ or 8-dimensional $(p = 2)$ extension. Therefore spacetime and imbedding space dimensions of TGD emerge very naturally in p-adic context.

The definition of pseudo-Riemannian metric poses problem: it seems that one should be able to make distinction between negative and positive p-adic numbers. A possible manner to make this distinction is to p-adic numbers with unit norm to be positive or negative according to whether they are squares or not. This definition makes sense if -1 does not possess square root: this is true for $p\ mod\ 4 = 3$. This condition will be encountered in most applications of p-adic numbers. At analytic level the definition generalizes

in obvious manner: what is required that the components of the metric are p-adically real numbers. The p-adic counter part of the Minkowski metric can be defined as

$$ds_p^2 = (dm^0)^2 - ((dm^1)^2 + (dm^2)^2 + (dm^3)^2) \tag{49}$$

The real image of this line element under canonical identification is nonnegative but metric allows to define the p-adic counterpart of M^4 lightcone as the surface $(m^0)^2 - ((m^1)^2 + (m^2)^2 + (m^3)^2) = 0$ and this surface can be regarded as a fractal counterpart of the ordinary light cone. Furthermore, this metric allows the p-adic counterpart of Lorentz group as its group of symmetries.

An interesting possibility is that one could define the length of a fractal curve (coast line of Britain) using p-adic Riemannian geometry. A possible model of this curve is obtained by identifying ordinary real plane with its p-adic counterpart via canonical identification and modelling the fractal curve with p-adically continuous or even analytic curve $x = x(t)$. The real counterpart of this curve is certainly fractal and need not have well defined length. The p-adic length of this curve can be defined as p-adic integral of $s_p = \int ds$ and its real counterpart s_R obtained by canonical identification can be defined to be the real length of the curve.

The concept of p-adic Riemann manifold as such is not quite enough for the mathematization of the topological condensate concept. Rather, topological condensate can be regarded as a surface obtained by glueing together p-adic spacetime regions with different values of p together along their boundaries. Each region is regarded as submanifold p-adic counterpart of $H = M^4_+ \times CP_2$. A natural manner to perform the gluing operation is to use canonical identification to map the boundaries of two regions p_1 and p_2 to real imbedding space H and to require that p_1 and p_2 boundary points correspond to same point in H.

8.2 p-Adic conformal geometry

It would be nice to have a generalization of ordinary conformal geometry to p-adic context. The following considerations and results of p-adic TGD suggest that the induced Kähler form defining Maxwell field on spacetime surface could be the basic entity of 4-dimensional conformal geometry rather

than metric. If the existence of square root is required the dimension of this geometry is $D = 4$ of $D = 8$ depending on the value of p. In the following it is assumed that the extension used is the minimal extension allowing square root and $p \ mod \ 4 = 3$ condition holds so that imaginary unit belongs to the generators of the extension.

In 2-dimensional case line element transforms by a conformal scale factor in p-analytic map $Z \rightarrow f(Z)$. In four-dimensional case this requirement leads to degenerate line element

$$\begin{aligned} ds^2 &= g(Z, Z_c, ...)dZdZ_c \\ &= g(Z, Z_c, ..)(dx^2 + dy^2 + p(du^2 + dv^2) + 2\sqrt{p}(dxdu + dydv)) \end{aligned} \quad (50)$$

where the conformal factor $g(Z, Z_c, ..)$ is invariant under complex conjugation. The metric tensor associated with the line element does not possess inverse. This is obvious from the fact that line element depends on two coordinates Z, Z_c only so that p-adic conformal metric is effectively 2-dimensional rather than 4-dimensional. It therefore seems that one must give up conformal covariance requirement for line element.

In two-dimensional conformal geometry angles are simplest conformal invariants and are expressible in terms of the inner product. In 4-dimensional case one can define invariants, which are analogous to angles. Let A and B be two vectors in 4-dimensional quadratic extension allowing square root. Denote A (B) and its various conjugates by A_i (B_i), $i = 1, 2, 3, 4$. Define phase like quantities $X_{ij} = exp(i2\Phi_{ij})$ between A and B by the following formulas

$$X_{ij} \equiv \frac{A_i A_j B_k B_l}{\sqrt{A_1 A_2 A_3 A_4}\sqrt{B_1 B_2 B_3 B_4}} \quad (51)$$

where i, j, k, l is permutation of $1, 2, 3, 4$. Each quantity X_{ij} is invariant under one of the conjugations $_c$, $\hat{}$ or $\hat{}_c$ and X_{ij} has values in 2-dimensional subspace of the 4-dimensional extension. As in ordinary case the angles are invariant under conjugation and this means that only 3 angle like quantities exists: this is in accordance with the fact that 3-angles are needed to specify the orientation of the vector A with respect to the vector B.

One can define also more general invariants using four vectors A, B, C, D and permutations i, j, k, l and r, s, t, u of $1, 2, 3, 4$

$$
\begin{aligned}
U_{ijkl} &= \frac{X_{ijkl}}{X_{rstu}} \\
X_{ijkl} &\equiv A_i B_j C_k D_l
\end{aligned}
\qquad (52)
$$

The number of the functionally independent invariants is reduced if various conjugates of invariants are not counted as different invariants. If 2 or 3 vectors are identical one obtains as special case invariants associated with 3 and 2 vectors. If there are only two vectors the number of the functionally independent invariants is 6.

There exists quadratic conformal covariants associated with tensors of weight two. The general form of the covariant is given by

$$
X = g^{ij:kl} A_{ij} B_{kl} \qquad (53)
$$

The tensor $g^{ij:kl}$ has the property that in complex coordinates $Z, \bar{Z}, \hat{Z}, \bar{\hat{Z}}$ the only nonvanishing components of the tensor have $i \neq j \neq k \neq l$. This guarantees multiplicative transformation property in conformal transformations $Z \rightarrow W(Z)$:

$$
X(W) = \frac{dW}{dZ} \frac{d\bar{W}}{d\bar{Z}} \frac{d\hat{W}}{d\hat{Z}} \frac{d\bar{\hat{W}}}{d\bar{\hat{Z}}} X(Z) \qquad (54)
$$

The simplest example of tensor $g^{ij:kl}$ is permutation symbol and the instanton density of any gauge field defines p-adic conformal covariant (the quantity is actually $Diff^4$ invariant).

The Kähler form of CP_2 is self dual but this property in general does not hold true for the induced Kähler form defining Maxwell field on spacetime surface. Kähler action density (Maxwell action) formed formed from the induced Kähler form on spacetime surface is in general not p-adic conformal invariant as such whereas the 'instanton density' is conformal invariant. It turns out that if CP_2 complex coordinate (4-dimensional extension) is p-adically analytic function of M^4 complex coordinate then the induced Kähler

form is self dual in the approximation that the induced metric is flat and one can express Kähler action density as

$$J^{\alpha\beta}J_{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}J_{\alpha\beta}J_{\gamma\delta} \tag{55}$$

This quantity satisfies the conditions guaranteing multiplicative transformation property under p-adic conformal transformations. What is nice that p-dically analytic maps define approximate extremals of Kähler action: action density however vanishes identically in flat metric approximation. An interesting open problem is whether one could find more general extremals of Kähler action satisfying the condition

$$J^{\alpha\beta} = g^{\alpha\beta\gamma\delta}J_{\gamma\delta} \tag{56}$$

such that the tensor $g^{\cdots}$ satisfies the required conditions but does not reduce to the permutation symbol.

Whether 4-dimensional p-adic conformal invariance plays role in p-adic TGD is not clear. It turns out the entire $Diff(M^4)$ rather than only $Conf(M^4)$ acts as approximate symmetries of Kähler action (broken only by gravitational effects) and that it is this larger invariance, which seems to be relevant for the dynamics of the interior of spacetime surface. It is p-adic counterpart of the ordinary 2-dimensional conformal invariance on boundary components of 3-surface, which plays key role in the calculation of particle masses.

9 p-Adic symmetries

The most basic level questions physicist can ask about p-adic numbers are related to symmetries. It seems obvious that the concept of Lie-group generalizes: nothing prevents from replacing the real or complex representation spaces associated with the definitions of classical Lie-groups with linear space associated with some algebraic extension of p-adic numbers: the defining algebraic conditions, such as unitarity or orthogonality properties, make sense for algebraically extended p-adic numbers, too. In case of orthogonal groups one must replace the ordinary real inner product with p-adically real inner

product $\sum_k X_k^2$ in a Cartesian power of a purely real extension of p-adic numbers: it should be emphasized that this inner product must be p-adic valued. In the unitary case one must consider complexification of a Cartesian power of purely real extension with inner produc $\sum \bar{Z}_k Z_k$. It should be emphasized however that the p-adic inner product differs from the ordinary one so that the action of, say, p-adic counterpart of rotation group in R_p^3 induces in R^3 an action, which need not have much to do with ordinary rotations so that the generalization is physically highly nontrivial. For very large values of p there are however good reasons to expect that locally the action of these groups resembles the action of their real counter parts.

A simple example is provided by the generalization of rotation group $SO(2)$. The rows of a rotation matrix are in general n orthonormalized vectors with the property that the components of these vectors have p-adic norm not larger than one. In case of $SO(2)$ this means the the matrix elements $a_{11} = a_{22} = a, a_{12} = -a_{21} = b$ satisfy the conditions

$$\begin{aligned} a^2 + b^2 &= 1 \\ |a|_p &\leq 1 \\ |b|_p &\leq 1 \end{aligned} \quad (57)$$

One can formally solve a as $a = \sqrt{1-b^2}$ but the solution doesn't exists always. There are various possibilities to define the orthogonal group.
a) One possibility is to allow only those values of a for which square root exists as p-adic number. In case of orthogonal group this requires that both $b = sin(\Phi)$ and $a = cos(\Phi)$ exist as p-adic numbers. If one requires rurther that a and b make sense also as ordinary rational numbers, they define Pythagorean triangle with integer sides and the group becomes discrete and cannot be regarded as Lie-group. Pythagorean triangles emerge for any rational counterpart of any Lie-group.
b) Other possibility is to allow an extension of p-adic numbers allowing square root. The minimal extensions has dimension 4 (8) for $p > 2$ ($p = 2$). Therefore spacetime dimension and imbedding space dimension emerge naturally as minimal dimensions for spaces, where p-adic $SO(2)$ acts 'stably'. The requirement that a and b are real is necessary unless one wants complexification of the $so(2)$ and gives constraints on the values of group parameters and again Lie-group property is expected to be lost.

c) Lie-group property is guaranteed if the allowed group elements are expressible as exponents of Lie-algebra generator Q. $g(t) = exp(iQt)$. This exponents exists only provided the p-adic norm of t is smaller than one. If one uses square root allowing extension one can require that t satisfies $|t| \leq p^{-n/2}$, $n > 0$ and one obtains a decreasing hierarchy of groups $G_1, G_2, ...$ For physically interesting values of p (typically of order $p = 2^{127} - 1$) the real counterparts of the transformations of these groups are extremely near to unit element of group. These conclusions hold true for any group. An especially interesting example physically is the group of 'small' Lorentz transformations with $t = O(\sqrt{p})$. If the rest energy of the particle is of order $O(\sqrt{p})$: $E_0 = m = m_0\sqrt{p}$ (as it turns out) then the Lorentz boost with velocity $\beta = \beta_0\sqrt{p}$ gives particle with energy $E = m/\sqrt{1-\beta_0^2 p} = m(1 + \frac{\beta_0^2 p}{2} + ..)$ so that $O(p^{1/2})$ term in energy is Lorentz invariant. This suggests that non-relativistic regime corresponds to small Lorentz transformations whereas in genuinely relativistic regime one must include also the discrete group of 'large' Lorentz transformation with rational transformations matrices.

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